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Abstract

In this paper, convex interval games are introduced and some characterizations are given. Some economic situations leading to convex interval games are discussed. The Weber set and the Shapley value are defined for a suitable class of interval games and their relations with the interval core for convex interval games are established. A square operator is introduced which allows us to obtain interval solutions starting from classical cooperative game theory solutions. It turns out that on the class of convex interval games the square Weber set coincides with the interval core.

JEL Classification: C71

Keywords: cooperative games, interval data, convex games, the core, the Weber set, the Shapley value

1 Introduction

In classical cooperative game theory convex games (Shapley (1971)) play an important role. Many characterizations and applications of classical convex games are available in the literature (Driessen (1988), Biswas et al. (1999),

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Branzei, Dimitrov and Tijs (2004, 2005, 2008), Martinez-Legaz (2006)). We extend here some notions and results for classical convex games to convex interval games. In the following we recall some definitions and results concerning classical cooperative game theory.

A *cooperative game* in coalitional form is an ordered pair $< N, v >$, where $N = \{1, 2, ..., n\}$ is the set of players, and $v : 2^N \to \mathbb{R}$ is a map, assigning to each coalition $S \in 2^N$ a real number, such that $v(\emptyset) = 0$. Often, we also refer to such a game as a TU (transferable utility) game. We denote by G^N the family of all classical cooperative games with player set N. The core (Gillies (1959)) is a central solution concept on G^N. The core $C(v)$ of $v \in G^N$ is defined by

$$C(v) = \left\{ x \in \mathbb{R}^N | \sum_{i \in N} x_i = v(N); \sum_{i \in S} x_i \geq v(S) \text{ for each } S \in 2^N \right\}.$$

A game $v \in G^N$ is convex if and only if the supermodularity condition $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for each $S, T \in 2^N$ holds true. In this paper we also refer to such convex games as supermodular games and use the following characterizations of classical convex games.

Theorem 1.1. *(Theorem 4.9 in Branzei, Dimitrov and Tijs (2005)) Let $v \in G^N$. The following five assertions are equivalent:*

(i) *$< N, v >$ is convex;*

(ii) *For all $S_1, S_2, U \in 2^N$ with $S_1 \subset S_2 \subset N \setminus U$ we have*

$$v(S_1 \cup U) - v(S_1) \leq v(S_2 \cup U) - v(S_2);$$

(iii) *For all $S_1, S_2 \in 2^N$ and $i \in N$ such that $S_1 \subset S_2 \subset N \setminus \{i\}$ we have*

$$v(S_1 \cup \{i\}) - v(S_1) \leq v(S_2 \cup \{i\}) - v(S_2);$$

(iv) *Each marginal vector $m^\sigma(v)$ of the game v with respect to the permutation σ belongs to the core $C(v)$;*

(v) *$W(v) = C(v)$, where $W(v)$ is the Weber set (Weber (1988)) of v which is defined as the convex hull of the marginal vectors of v.*

Notice that Theorem 1.1 implies that convex games have a nonempty core. In Section 5 we refer to a game $< N, v >$ whose core is nonempty as a balanced game. On the class of convex games solution concepts have nice properties. We recall that the Shapley value of a convex game belongs to the core of the game and the core is the unique stable set of the game. Also, the core is an additive map on the class of convex games (Dragan, Potters and Tijs (1989)). For details regarding special properties of solution concepts on the class of conex games we refer the reader to Branzei, Dimitrov and Tijs (2005).

We recall that a *cooperative interval game* in coalitional form (Alparslan Gök, Miquel and Tijs (2008)) is an ordered pair $< N, w >$ where $N = \{1, 2, \ldots, n\}$ is the set of players, and $w : 2^N \to I(\mathbb{R})$ is the characteristic function such that $w(\emptyset) = [0, 0]$, where $I(\mathbb{R})$ is the set of all closed intervals in $\mathbb{R}$. For each $S \in 2^N$, the worth set (or worth interval) $w(S)$ of the coalition S in the interval game $< N, w >$ is of the form $[\underline{w}(S), \overline{w}(S)]$, where $\underline{w}(S)$ is the lower bound and $\overline{w}(S)$ is the upper bound of $w(S)$. We denote by IG^N the family of all interval games with player set N. Note that if all the worth intervals are degenerate intervals, i.e. $\underline{w}(S) = \overline{w}(S)$ for each $S \in 2^N$, then the interval game $< N, w >$ corresponds in a natural way to the classical cooperative game $< N, v >$ where $v(S) = \underline{w}(S)$.
The paper is organized as follows. In Section 2 we recall basic notions and facts from the theory of cooperative interval games. In Section 3 we introduce supermodular and convex interval games and give some characterizations of convex interval games. Some economic situations leading to convex interval games are briefly discussed. In Section 4 we introduce for size monotonic interval games the notions of marginal operators, the Shapley value and the Weber set and study their properties for convex interval games. In Section 5 we introduce the square operator and describe some interval solutions for interval games that have close relations with existing solutions from the classical cooperative game theory. It turns out that on the class of convex interval games the interval core and the square Weber set coincide. Finally, in Section 6 we conclude with some remarks on further research.

2 Preliminaries on interval calculus and interval games

In this section some preliminaries from interval calculus and some useful results from the theory of cooperative interval games are given (Alparslan Gök, Branzei and Tijs (2008a)).
Let $I, J \in I(\mathbb{R})$ with $I = \left[\underline{I}, \overline{I}\right]$, $J = \left[\underline{J}, \overline{J}\right]$, $|I| = \overline{I} - \underline{I}$ and $\alpha \in \mathbb{R}_+$. Then,

(i) $I + J = \left[\underline{I}, \overline{I}\right] + \left[\underline{J}, \overline{J}\right] = \left[\underline{I} + \underline{J}, \overline{I} + \overline{J}\right]$;

(ii) $\alpha I = \alpha \left[\underline{I}, \overline{I}\right] = \left[\alpha \underline{I}, \alpha \overline{I}\right]$.

By (i) and (ii) we see that $I(\mathbb{R})^N$ has a cone structure.
In this paper we also need a partial substraction operator. We define $I - J$, only if $|I| \geq |J|$, by $I - J = \left[\underline{I}, \overline{I}\right] - \left[\underline{J}, \overline{J}\right] = \left[\underline{I} - \underline{J}, \overline{I} - \overline{J}\right]$. Note that $\underline{I} - \underline{J} \leq \overline{I} - \overline{J}$. We recall that I is weakly better than J, which we denote by $I \succcurlyeq J$, if and only if $\underline{I} \geq \underline{J}$ and $\overline{I} \geq \overline{J}$. We also use the reverse notation $J \preccurlyeq I$, if and only if $\underline{J} \leq \underline{I}$ and $\overline{J} \leq \overline{I}$. We say that I is better than J, which we denote by $I \succ J$, if and only if $I \succcurlyeq J$ and $I \neq J$.
For $w_1, w_2 \in IG^N$ we say that $w_1 \preccurlyeq w_2$ if $w_1(S) \preccurlyeq w_2(S)$, for each $S \in 2^N$.
For $w_1, w_2 \in IG^N$ and $\lambda \in \mathbb{R}_+$ we define $< N, w_1 + w_2 >$ and $< N, \lambda w >$ by $(w_1 + w_2)(S) = w_1(S) + w_2(S)$ and $(\lambda w)(S) = \lambda \cdot w(S)$ for each $S \in 2^N$. So, we conclude that IG^N endowed with $\preccurlyeq$ is a partially ordered set and has a cone structure with respect to addition and multiplication with non-negative scalars described above. For $w_1, w_2 \in IG^N$ with $|w_1(S)| \geq |w_2(S)|$ for each $S \in 2^N$, $< N, w_1 - w_2 >$ is defined by $(w_1 - w_2)(S) = w_1(S) - w_2(S)$.
Now, we recall that the *interval core* $\mathcal{C}(w)$ of the interval game w is defined by

$$\mathcal{C}(w) = \left\{ (I_1, \ldots, I_n) \in I(\mathbb{R})^N | \sum_{i \in N} I_i = w(N), \sum_{i \in S} I_i \succcurlyeq w(S), \text{ for all } S \in 2^N \setminus \{\emptyset\} \right\}.$$

A game $w \in IG^N$ is called $\mathcal{I}$*-balanced* if for each balanced map $\lambda : 2^N \setminus \{\emptyset\} \to \mathbb{R}_+$ we have $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) w(S) \preccurlyeq w(N)$. Note that a map $\lambda : 2^N \setminus \{\emptyset\} \to \mathbb{R}_+$ is called a balanced map (Tijs (2003)) if $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) e^S = e^N$. A game $w \in IG^N$ is $\mathcal{I}$-balanced if and only if $\mathcal{C}(w) \neq \emptyset$ (Theorem 3.1 in Alparslan Gök, Branzei and Tijs (2008a)). We denote by $\mathcal{I}BIG^N$ the class of $\mathcal{I}$-balanced interval games with player set N.

Let $w \in IG^N$, $I = (I_1, \ldots, I_n), J = (J_1, \ldots, J_n) \in \mathcal{I}(w)$ and $S \in 2^N \setminus \{\emptyset\}$. We say that I dominates J via coalition S, denoted by $I \operatorname{dom}_S J$, if

(i) $I_i \succ J_i$ for all $i \in S$,

(ii) $\sum_{i \in S} I_i \preccurlyeq w(S)$.

For $S \in 2^N \setminus \{\emptyset\}$ we denote by $D(S)$ the set of those elements of $\mathcal{I}(w)$ which are dominated via S. I is called undominated if there does not exist J and a coalition S such that $J \operatorname{dom}_S I$. The *interval dominance core* $\mathcal{DC}(w)$ of $w \in IG^N$ consists of all undominated elements in $\mathcal{I}(w)$, i.e. it is the complement in $\mathcal{I}(w)$ of $\cup \left\{D(S) | S \in 2^N \setminus \{\emptyset\}\right\}$.

3 Supermodular and convex interval games

In this section some classical TU-games associated with an interval game $w \in IG^N$ will play a key role, namely the *border* games $< N, \underline{w} >$, $< N, \overline{w} >$ and the *length* game $< N, |w| >$, where $|w|(S) = \overline{w}(S) - \underline{w}(S)$ for each $S \in 2^N$. Note that $\overline{w} = \underline{w} + |w|$.
We say that a game $< N, w >$ is *supermodular* if

$$w(S) + w(T) \preccurlyeq w(S \cup T) + w(S \cap T) \text{ for all } S, T \in 2^N. \quad (1)$$

We introduce the notion of convex interval game and denote by CIG^N the class of convex interval games with player set N. We call a game $w \in IG^N$ *convex* if $< N, w >$ is supermodular and its length game $< N, |w| >$ is also supermodular. Note that the nonempty set CIG^N is a subcone of IG^N.
The next proposition shows that traditional convex games can be embedded in a natural way in the class of convex interval games.

Proposition 3.1. *If $v \in G^N$ is convex then the corresponding game $w \in IG^N$ which is defined by $w(S) = [v(S), v(S)]$ for each $S \in 2^N$ is also convex.*

The proof of this proposition is left to the reader. In the next proposition we give some characterizations of supermodular and convex games $w \in IG^N$ based on their related length game $|w| \in G^N$ and border games $\underline{w}, \overline{w} \in G^N$.

Proposition 3.2. *Let $w \in IG^N$ and its related games $|w|, \underline{w}, \overline{w} \in G^N$. Then the following assertions hold:*

(i) *A game* $< N, w >$ *is supermodular if and only if its border games* $< N, \underline{w} >$ *and* $< N, \overline{w} >$ *are convex;*

(ii) *A game* $< N, w >$ *is convex if and only if its length game* $< N, |w| >$ *and its border games* $< N, \underline{w} >$, $< N, \overline{w} >$ *are convex;*

(iii) *A game* $< N, w >$ *is convex if and only if its border game* $< N, \underline{w} >$ *and the game* $< N, \overline{w} - \underline{w} >$ *are convex.*

Proof. (i) This assertion follows from formula (1).

(ii) By definition $< N, w >$ is convex if and only if $< N, w >$ and $< N, |w| >$ are both supermodular. By (i), $< N, w >$ is supermodular if and only if its border games are convex. Now, since supermodularity of $< N, |w| >$ is equivalent with its convexity, we conclude that $< N, w >$ is convex if and only if $< N, \underline{w} >$, $< N, \overline{w} >$ and $< N, |w| >$ are convex.

(iii) This assertion follows easily from (ii) by noting that $< N, |w| >$, $< N, \underline{w} >$ and $< N, \overline{w} >$ are convex if and only if $< N, \overline{w} - \underline{w} >$ and $< N, \underline{w} >$ are convex.

$\square$

The next example shows that a supermodular interval game is not necessarily convex.

EXAMPLE 3.1. *Let* $< N, w >$ *be the two-person interval game with* $w(\emptyset) = [0, 0]$, $w(1) = w(2) = [0, 1]$ *and* $w(1, 2) = [3, 4]$. *Here,* $< N, w >$ *is supermodular and the border games are convex, but* $|w|(1) + |w|(2) > |w|(1, 2) + |w|(\emptyset)$. *Hence,* $< N, w >$ *is not convex.*

The next example shows that an interval game whose length game is supermodular is not necessarily convex.

EXAMPLE 3.2. *Let* $< N, w >$ *be the three-person interval game with* $w(i) = [1, 1]$ *for each* $i \in N$, $w(N) = w(1, 3) = w(1, 2) = w(2, 3) = [2, 2]$ *and* $w(\emptyset) = [0, 0]$. *Here,* $< N, w >$ *is not convex, but* $< N, |w| >$ *is supermodular, since* $|w|(S) = 0$, *for each* $S \in 2^N$.

Interesting examples of convex interval games are unanimity interval games. First, we recall the definition of such games. Let $J \in I(\mathbb{R})$ with

$J \succcurlyeq [0,0]$ and let $T \in 2^N \setminus \{\emptyset\}$. The unanimity interval game based on J and T is defined by

$$u_{T,J}(S) = \begin{cases} J, & T \subset S \\ [0,0], & \textit{otherwise}, \end{cases}$$

for each $S \in 2^N$.
Clearly, $< N, |u_{T,J}| >$ is supermodular. The supermodularity of $< N, u_{T,J} >$ can be checked by considering the following case study:

	$u_{T,J}(A \cup B)$	$u_{T,J}(A \cap B)$	$u_{T,J}(A)$	$u_{T,J}(B)$
$T \subset A, T \subset B$	J	J	J	J
$T \subset A, T \not\subset B$	J	$[0,0]$	J	$[0,0]$
$T \not\subset A, T \subset B$	J	$[0,0]$	$[0,0]$	J
$T \not\subset A, T \not\subset B$	J or $[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$.

We call a game $< N, w >$ *size monotonic* if $< N, |w| >$ is monotonic, i.e. $|w|(S) \leq |w|(T)$ for all $S, T \in 2^N$ with $S \subset T$. For further use we denote by $SMIG^N$ the class of size monotonic interval games with player set N.

REMARK 3.1. *For size monotonic games $< N, w >$, $w(T) - w(S)$ is well defined for all $S, T \in 2^N$ with $S \subset T$ since $|w(T)| = |w|(T) \geq |w|(S) = |w(S)|$.*

REMARK 3.2. *Note that the fact that $< N, |w| >$ is supermodular implies that $< N, |w| >$ is monotonic because for each $S, T \in 2^N$ with $S \subset T$ we have*

$$|w|(T) + |w|(\emptyset) \geq |w|(S) + |w|(T \setminus S),$$

and from this inequality follows $|w|(S) \leq |w|(T)$ since $|w|(T \setminus S) \geq 0$.
As a by product we obtain that each game $w \in CIG^N$ is size monotonic.

For convex TU-games various characterizations are known. In the next theorem we give three characterizations of convex interval games inspired by Shapley (1971) .

THEOREM 3.1. *Let $w \in IG^N$ be such that $|w| \in G^N$ is supermodular. Then, the following three assertions are equivalent:*

(i) *$w \in IG^N$ is convex;*

(ii) *For all* $S_1, S_2, U \in 2^N$ *with* $S_1 \subset S_2 \subset N \setminus U$ *we have*

$$w(S_1 \cup U) - w(S_1) \preccurlyeq w(S_2 \cup U) - w(S_2); \quad (2)$$

(iii) *For all* $S_1, S_2 \in 2^N$ *and* $i \in N$ *such that* $S_1 \subset S_2 \subset N \setminus \{i\}$ *we have*

$$w(S_1 \cup \{i\}) - w(S_1) \preccurlyeq w(S_2 \cup \{i\}) - w(S_2).$$

Proof. We show (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (i).
Suppose that (i) holds. To prove (ii) take $S_1, S_2, U \in 2^N$ with $S_1 \subset S_2 \subset N \setminus U$. From (1) with $S_1 \cup U$ in the role of S and S_2 in the role of T we obtain (2) by noting that $S \cup T = S_2 \cup U$, $S \cap T = S_1$. Hence, (i) implies (ii).
That (ii) implies (iii) is straightforward (take $U = \{i\}$).
Now, suppose that (iii) holds. To prove (i) take $S, T \in 2^N$. Clearly, (1) holds if $S \subset T$. Suppose that $T \setminus S$ consists of the elements $i_1, \dots, i_k$ and let $D = S \cap T$. Then, from (iii) follows that

$$\begin{aligned}
w(S) - w(S \cap T) &= w(D \cup \{i_1\}) - w(D) \\
&+ \sum_{s=2}^{k} w(D \cup \{i_1, \dots, i_s\}) - w(D \cup \{i_1, \dots, i_{s-1}\}) \\
&\preccurlyeq w(T \cup \{i_1\}) - w(T) \\
&+ \sum_{s=2}^{k} w(T \cup \{i_1, \dots, i_s\}) - w(T \cup \{i_1, \dots, i_{s-1}\}) \\
&= w(S \cup T) - w(T), \text{ for each } S \in 2^N.
\end{aligned}$$

□

Next we give an example with an economic flavour leading to a convex interval game.

EXAMPLE 3.3. *Let* $N = \{1, 2, \dots, n\}$ *and let* $f : [0, n] \to I(\mathbb{R})$ *be such that* $f(x) = [f_1(x), f_2(x)]$ *for each* $x \in [0, n]$ *and* $f(0) = [0, 0]$. *Suppose that* $f_1 : [0, n] \to \mathbb{R}$, $f_2 : [0, n] \to \mathbb{R}$ *and* $(f_2 - f_1) : [0, n] \to \mathbb{R}$ *are convex monotonic increasing functions. Then, we can construct a corresponding interval game* $w : 2^N \to I(\mathbb{R})$ *such that* $w(S) = f(|S|) = [f_1(|S|), f_2(|S|)]$ *for each* $S \in 2^N$. *It is easy to show that* w *is a convex interval game with the*

symmetry property $w(S) = w(T)$ for each $S, T \in 2^N$ with $|S| = |T|$.
We can see $< N, w >$ as a production game if we interpret $f(s)$ for $s \in N$ as the interval reward which s players in N can produce by working together.

Before closing this section we indicate some other economic situations related to convex interval games. It is well known that classical public good situations (Moulin (1988)) and sequencing situations (Curiel, Pederzoli and Tijs (1989)) lead to convex games. In case of interval uncertainty in such situations under restricting conditions, convex interval games arise. Also, special bankruptcy situations (O'Neill (1982), Aumann and Maschler (1985) and Curiel, Maschler and Tijs (1987)) when the estate of the bank and the claims are intervals give rise in a natural way to convex interval games (Alparslan Gök, Branzei and Tijs (2008b)). Furthermore, airport situations (Littlechild and Owen (1977)) with interval data lead to concave interval games. An interval game $< N, w >$ is called *concave* if $< N, w >$ and $< N, |w| >$ are submodular, i.e. $w(S) + w(T) \succcurlyeq w(S \cup T) + w(S \cap T)$ and $|w|(S) + |w|(T) \geq |w|(S \cup T) + |w|(S \cap T)$, for all $S, T \in 2^N$.

4 The Shapley value and the Weber set

In this section we introduce marginal operators on the class of size monotonic interval games, define the Shapley value and the Weber set on this class of games, and study their properties on the class of convex interval games.
Denote by $\Pi(N)$ the set of permutations $\sigma : N \to N$. Let $w \in SMIG^N$. We introduce the notions of *interval marginal operator* corresponding to σ, denoted by m^σ, and of *interval marginal vector* of w with respect to σ, denoted by $m^\sigma(w)$. The marginal vector $m^\sigma(w)$ corresponds to a situation, where the players enter a room one by one in the order $\sigma(1), \sigma(2), \ldots, \sigma(n)$ and each player is given the marginal contribution he/she creates by entering. If we introduce the set $P_\sigma(i)$ of predecessors of i in σ by $P_\sigma(i) = \{r \in N | \sigma^{-1}(r) < \sigma^{-1}(i)\}$ where $\sigma^{-1}(i)$ denotes the entrance number of player i, then $m^\sigma_{\sigma(k)}(w) = w(P_\sigma(\sigma(k)) \cup \{\sigma(k)\}) - w(P_\sigma(\sigma(k)))$, or $m^\sigma_i(w) = w(P_\sigma(i) \cup \{i\}) - w(P_\sigma(i))$.
The following example illustrates that for interval games which are not size monotonic it might happen that some interval marginal vectors do not exist.

EXAMPLE 4.1. *Let $< N, w >$ be the interval game with $N = \{1, 2\}$, $w(1) = [1, 3], w(2) = [0, 0]$ and $w(1, 2) = [2, 3\frac{1}{2}]$. This game is not size monotonic.*

Note that $m^{(12)}(w)$ is not defined because $w(1,2) - w(1)$ is undefined since $|w(1,2)| < |w(1)|$.

THEOREM 4.1. *Let $w \in IG^N$. Then, the following assertions are equivalent:*

(i) *w is convex;*

(ii) *$|w|$ is supermodular and $m^\sigma(w) \in \mathcal{C}(w)$ for all $\sigma \in \Pi(N)$.*

Proof. (i) $\Rightarrow$ (ii) Let $w \in CIG^N$, let $\sigma \in \Pi(N)$ and take $m^\sigma(w)$. Clearly, $\sum_{k \in N} m_k^\sigma(w) = w(N)$. To prove that $m^\sigma(w) \in \mathcal{C}(w)$ we have to show that for $S \in 2^N$, $\sum_{k \in S} m_k^\sigma(w) \succcurlyeq w(S)$. Let $S = \{\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k)\}$ with $i_1 < i_2 < \ldots < i_k$. Then,

$$\begin{aligned}
w(S) &= w(\sigma(i_1)) - w(\emptyset) \\
&+ \sum_{r=2}^{k} (w(\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_r)) - w(\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_{r-1}))) \\
&\preccurlyeq w(\sigma(1), \ldots, \sigma(i_1)) - w(\sigma(1), \ldots, \sigma(i_1 - 1)) \\
&+ \sum_{r=2}^{k} (w(\sigma(1), \sigma(2), \ldots, \sigma(i_r)) - w(\sigma(1), \sigma(2), \ldots, \sigma(i_r - 1))) \\
&= \sum_{r=1}^{k} m_{\sigma(i_r)}^\sigma(w) = \sum_{k \in S} m_k^\sigma(w),
\end{aligned}$$

where the inequality follows from Theorem 3.1 (iii) applied to $i = \sigma(i_r)$ and

$$S_1 = \{\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_{r-1})\} \subset S_2 = \{\sigma(1), \sigma(2), \ldots, \sigma(i_{r-1})\}$$

for $r \in \{1, 2, \ldots, k\}$. Further, $|w|$ is supermodular.
(ii) $\Rightarrow$ (i) From $m^\sigma(w) \in \mathcal{C}(w)$ for all $\sigma \in \Pi(N)$ follows that $m^\sigma(\underline{w}) \in C(\underline{w})$ and $m^\sigma(\overline{w}) \in C(\overline{w})$ for all $\sigma \in \Pi(N)$. Now, by Theorem 1.1 we obtain that $< N, \underline{w} >$ and $< N, \overline{w} >$ are convex games. Since $< N, |w| >$ is convex by hypothesis, we obtain by Proposition 3.2 (ii) that $< N, w >$ is convex. $\square$

Now, we straightforwardly extend for size monotonic interval games two important solution concepts in cooperative game theory which are based on marginal worth vectors: the Shapley value (Shapley (1953)) and the Weber set (Weber (1988)).
The *interval Weber set* $\mathcal{W}$ on the class of size monotonic interval games is

defined by $\mathcal{W}(w) = conv\,\{m^\sigma(w)|\sigma \in \Pi(N)\}$ for each $w \in SMIG^N$. We notice that for traditional TU-games we have $W(v) \neq \emptyset$ for all $v \in G^N$, while for interval games it might happen that $\mathcal{W}(w) = \emptyset$ (in case none of the interval marginal vectors $m^\sigma(w)$ is defined). Clearly, $\mathcal{W}(w) \neq \emptyset$ for all $w \in SMIG^N$. Further, it is well known that $C(v) = W(v)$ if and only if $v \in G^N$ is convex (see Theorem 1.1). However, this result can not be extended to convex interval games as the following example illustrates.

Example 4.2. *Let $N = \{1,2\}$ and let $w : 2^N \to I(\mathbb{R})$ be defined by $w(1) = w(2) = [0,1]$ and $w(1,2) = [2,4]$. This game is convex. Further, $m^{(1,2)}(w) = ([0,1],[2,3])$ and $m^{(2,1)}(w) = ([2,3],[0,1])$, belong to the interval core $\mathcal{C}(w)$ and $\mathcal{W}(w) = conv\left\{m^{(1,2)}(w), m^{(2,1)}(w)\right\}$. Notice that $([\frac{1}{2},1\frac{3}{4}],[1\frac{1}{2},2\frac{1}{4}]) \in \mathcal{C}(w)$ and there is no $\alpha \in [0,1]$ such that $\alpha m^{(1,2)}(w) + (1-\alpha)m^{(2,1)}(w) = ([\frac{1}{2},1\frac{3}{4}],[1\frac{1}{2},2\frac{1}{4}])$. Hence, $\mathcal{W}(w) \subset \mathcal{C}(w)$ and $\mathcal{W}(w) \neq \mathcal{C}(w)$.*

Proposition 4.1. *Let $w \in CIG^N$. Then, $\mathcal{W}(w) \subset \mathcal{C}(w)$.*

Proof. By Theorem 4.1 we have $m^\sigma(w) \in \mathcal{C}(w)$ for each $\sigma \in \Pi(N)$. Now, we use the convexity of $\mathcal{C}(w)$. □

In Section 5 we introduce a new notion of Weber set and show that the equality between the interval core and that Weber set still holds on the class of convex interval games.

The *interval Shapley value* $\Phi : SMIG^N \to I(\mathbb{R})^N$ is defined by

$$\Phi(w) := \frac{1}{n!}\sum_{\sigma\in\Pi(N)} m^\sigma(w), \text{ for each } w \in SMIG^N. \quad (3)$$

Clearly, we can write (3) as follows

$$\Phi_i(w) = \frac{1}{n!}\sum_{\sigma\in\Pi(N)} (w(P^\sigma(i)\cup\{i\}) - w(P^\sigma(i))).$$

Since $\Phi(w) \in \mathcal{W}(w)$ for each $w \in SMIG^N$, by Proposition 4.1 we have $\Phi(w) \in \mathcal{C}(w)$ for each $w \in CIG^N$. Without going into details we note here that the Shapley value Φ on the class of convex interval games satisfies the properties of additivity, efficiency, symmetry and dummy player.

5 Interval solutions obtained with the square operator

In this section we introduce the square operator, which assigns to each pair $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ with $x \leq y$ an element of $I(\mathbb{R})^N$. For many classical solutions for TU-games one can define with the aid of this square operator a corresponding square solution on suitable classes of interval games.
Let $a, b \in \mathbb{R}^n$ with $a \leq b$. Then, a and b determine a hypercube

$$\mathcal{H} = \{x \in \mathbb{R}^n | a_i \leq x_i \leq b_i \text{ for each } i \in \{1, \ldots, n\}\}.$$

Here, $\mathcal{H}$ can be seen as the Cartesian product $I_1 \times \ldots \times I_n$ of the closed intervals $I_1, \ldots, I_n \in I(\mathbb{R})$ with $I_i = [a_i, b_i]$ for each $i \in \{1, \ldots, n\}$.
We denote by $a \Box b$ the vector $(I_1, \ldots, I_n) \in I(\mathbb{R})^n$ generated by the pair $(a, b) \in \mathbb{R}^n$, $a \leq b$. Let $A, B \subset \mathbb{R}^n$. Then, we denote by $A \Box B$ the subset of $I(\mathbb{R})^n$ of all elements of the form $([a_1, b_1], \ldots, [a_n, b_n])$ where $a = (a_1, \ldots, a_n) \in A$, $b = (b_1, \ldots, b_n) \in B$ and $a \leq b$. Shortly, $A \Box B = \{a \Box b | a \in A, b \in B, a \leq b\}$. Note that if $\{a \Box b | a \in A, b \in B, a \leq b\} = \emptyset$ then $A \Box B = \emptyset$.
Now, with the use of the $\Box$ operator, we give a procedure to extend classical solutions and multi-solutions on subsets of G^N, to interval solutions and interval multi-solutions on suitable subsets of IG^N.
Let $\psi : H^N \to \mathbb{R}^N$ be a solution for classical TU-games on $H^N \subset G^N$. We are interested in games $w \in IG^N$ for which $\underline{w}, \overline{w} \in H^N$ and $\psi(\underline{w}) \leq \psi(\overline{w})$. The class of such games we denote by IH^N_ψ. Now, we define $\psi^\Box : IH^N_\psi \to I(\mathbb{R})^N$ by $\psi^\Box(w) = \psi(\underline{w}) \Box \psi(\overline{w})$ for each $w \in IH^N_\psi$. So, $\psi^\Box(w) = \{x \Box y | x \in \psi(\underline{w}), y \in \psi(\overline{w}), x \leq y\}$ for $w \in IH^N_\psi$.
For a multi-solution $\mathcal{F} : H^N \to\to \mathbb{R}^N$ we proceed in a similar way and define $\mathcal{F}^\Box : IH^N_\mathcal{F} \to\to I(\mathbb{R})^N$ by $\mathcal{F}^\Box = \mathcal{F}(\underline{w}) \Box \mathcal{F}(\overline{w})$ for each $w \in IH^N_\mathcal{F}$.
Now, we focus on this procedure for some (multi) solutions such as the core, the marginal operators, the Shapley value and the Weber set on suitable classes of interval games. In the next proposition we connect the $\mathcal{I}$-balancedness of $< N, w >$ with the balancedness of its border games.

PROPOSITION 5.1. *If $< N, w >$ is $\mathcal{I}$-balanced then the border games $< N, \underline{w} >$ and $< N, \overline{w} >$ are balanced.*

Proof. Let $< N, w >$ be $\mathcal{I}$-balanced. Then, for each balanced map $\lambda : 2^N \setminus \{\emptyset\} \to \mathbb{R}_+$ we have $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) w(S) \preccurlyeq w(N)$ implying that

$\sum_{S\in 2^N\setminus\{\emptyset\}}\lambda(S)\underline{w}(S)\leq \underline{w}(N)$ and $\sum_{S\in 2^N\setminus\{\emptyset\}}\lambda(S)\overline{w}(S)\leq \overline{w}(N)$, which express the balancedness of the border games of w. $\square$

We define the square interval core $\mathcal{C}^{\square}:\mathcal{I}BIG^N\to\to I(\mathbb{R})^N$ by $\mathcal{C}^{\square}(w)=C(\underline{w})\square C(\overline{w})$ for each $w\in\mathcal{I}BIG^N$.

Proposition 5.2. *Let $w\in\mathcal{I}BIG^N$. Then, $\mathcal{C}(w)=\mathcal{C}^{\square}(w)$.*

Proof. $(I_1,\ldots,I_n)\in\mathcal{C}(w)$ if and only if $(\underline{I}_1,\ldots,\underline{I}_n)\in C(\underline{w})$ and $(\overline{I}_1,\ldots,\overline{I}_n)\in C(\overline{w})$ if and only if $(I_1,\ldots,I_n)=(\underline{I}_1,\ldots,\underline{I}_n)\square(\overline{I}_1,\ldots,\overline{I}_n)\in\mathcal{C}^{\square}(w)$. $\square$

Since $CIG^N\subset\mathcal{I}BIG^N$ we obtain that $\mathcal{C}(w)=\mathcal{C}^{\square}(w)$ for each $w\in CIG^N$.
Now, we notice that for each $w\in SMIG^N$ the interval marginal vectors $m^{\sigma}(w)$ are defined for each $\sigma\in\Pi(N)$, because the monotonicity of $|w|$ implies $\overline{w}(S\cup\{i\})-\underline{w}(S\cup\{i\})\geq\overline{w}(S)-\underline{w}(S)$, which can be rewritten as $\overline{w}(S\cup\{i\})-\overline{w}(S)\geq\underline{w}(S\cup\{i\}-\underline{w}(S)$. So, $w(S\cup\{i\})-w(S)$ is defined for each $S\subset N$ and $i\notin S$.
We define the square Weber set $\mathcal{W}^{\square}:SMIG^N\to\to I(\mathbb{R})^N$ by $\mathcal{W}^{\square}(w)=W(\underline{w})\square W(\overline{w})$ for each $w\in SMIG^N$. Note that $\mathcal{C}^{\square}(w)=\mathcal{W}^{\square}(w)$ if $w\in CIG^N$.

Proposition 5.3. *Let $w\in IG^N$ be such that $|w|\in G^N$ is supermodular and let $\sigma\in\Pi(N)$. Then, $m^{\sigma}(w)=m^{\sigma}(\underline{w})\square m^{\sigma}(\overline{w})$.*

Proof. By definition,

$$m^{\sigma}(\underline{w})=(\underline{w}(\sigma(1)),\underline{w}(\sigma(1),\sigma(2))-\underline{w}(\sigma(1)),\ldots,\underline{w}(\sigma(1),\ldots,\sigma(n))-\underline{w}(\sigma(1),\ldots,\sigma(n-1)),$$

and

$$m^{\sigma}(\overline{w})=(\overline{w}(\sigma(1)),\overline{w}(\sigma(1),\sigma(2))-\overline{w}(\sigma(1)),\ldots,\overline{w}(\sigma(1),\ldots,\sigma(n))-\overline{w}(\sigma(1),\ldots,\sigma(n-1)).$$

Now, we prove that $m^{\sigma}(\overline{w})-m^{\sigma}(\underline{w})\geq 0$. Since $|w|=\overline{w}-\underline{w}$ is a classical convex game we have for each $k\in N$

$$\begin{aligned}
m^{\sigma}_{\sigma(k)}(\overline{w})-m^{\sigma}_{\sigma(k)}(\underline{w}) &= (\overline{w}-\underline{w})(\sigma(1),\ldots,\sigma(k))-(\overline{w}-\underline{w})(\sigma(1),\ldots,\sigma(k-1))\\
&= |w|(\sigma(1),\ldots,\sigma(k))-|w|(\sigma(1),\ldots,\sigma(k-1))\\
&\geq |w|(\sigma(k))-|w|(\emptyset)=|w|(\sigma(k))\geq 0,
\end{aligned}$$

where the first inequality follows from the properties of classical convex games. So, $m^\sigma(\underline{w}) \leq m^\sigma(\overline{w})$, and

$$m^\sigma(\underline{w})\Box m^\sigma(\overline{w}) = (w(\sigma(1)), \ldots, w(\sigma(1), \ldots, \sigma(n)) - w(\sigma(1), \ldots, \sigma(n-1))) = m^\sigma(w).$$

$\Box$

Since $CIG^N \subset SMIG^N$ we obtain from Proposition 5.3 that $m^\sigma(w) = m^\sigma(\underline{w})\Box m^\sigma(\overline{w})$ for each $w \in CIG^N$ and $\sigma \in \Pi(N)$.
The next two theorems are very important because they extend for interval games, with the square interval Weber set in the role of the Weber set, the well known results in classical cooperative game theory that $C(v) \subset W(v)$ for each $v \in G^N$ (Weber (1988)) and $C(v) = W(v)$ if and only if v is convex (Ichiishi (1981)).

THEOREM 5.1. *Let $w \in SMIG^N$. Then, $\mathcal{C}(w) \subset \mathcal{W}^\Box(w)$.*

Proof. Let $(I_1, \ldots, I_n) \in \mathcal{C}(w)$. Then, by Proposition 5.2, $(\underline{I}_1, \ldots, \underline{I}_n) \in C(\underline{w})$ and $(\overline{I}_1, \ldots, \overline{I}_n) \in C(\overline{w})$, and, because $C(v) \subset W(v)$ for each $v \in G^N$, we obtain $(\underline{I}_1, \ldots, \underline{I}_n) \in W(\underline{w})$ and $(\overline{I}_1, \ldots, \overline{I}_n) \in W(\overline{w})$. Hence, we obtain $(I_1, \ldots, I_n) \in \mathcal{W}^\Box(w)$. $\Box$

From Theorem 5.1 and Proposition 4.1 we obtain that $\mathcal{W}(w) \subset \mathcal{W}^\Box(w)$ for each $w \in CIG^N$.

THEOREM 5.2. *Let $w \in \mathcal{I}BIG^N$. Then, the following assertions are equivalent:*

(i) *w is convex;*

(ii) *$|w|$ is supermodular and $\mathcal{C}(w) = \mathcal{W}^\Box(w)$.*

Proof. By Proposition 3.2 (ii), w is convex if and only if $|w|, \underline{w}$ and $\overline{w}$ are convex. Clearly, the convexity of $|w|$ is equivalent with its supermodularity. Further, by Theorem 1.1, $\underline{w}$ and $\overline{w}$ are convex if and only if $W(\underline{w}) = C(\underline{w})$ and $W(\overline{w}) = C(\overline{w})$. These equalities are equivalent with $\mathcal{W}^\Box(w) = \mathcal{C}^\Box(w)$. Finally, since w is $\mathcal{I}$-balanced by hypothesis, we have by Proposition 5.2 that $\mathcal{C}(w) = \mathcal{W}^\Box(w)$. $\Box$

Next we introduce the square Shapley value $\Phi^\Box$ on the class of games $w \in IG^N$ for which $|w|$ is supermodular by $\Phi^\Box(w) = \phi(\underline{w})\Box\phi(\overline{w})$. In the following proposition we prove that the interval Shapley value is equal to the square Shapley value on this class of interval games.

PROPOSITION 5.4. *Let $w \in IG^N$ be such that $|w|$ is supermodular. Then, $\Phi(w) = \phi(\underline{w})\Box\phi(\overline{w})$.*

Proof. From (3) and Proposition 5.3 we have

$$\Phi(w) = \frac{1}{n!}\sum_{\sigma\in\Pi(N)} m^\sigma(w) = \frac{1}{n!}\sum_{\sigma\in\Pi(N)} (m^\sigma(\underline{w})\Box m^\sigma(\overline{w})) \ =$$

$$\left(\frac{1}{n!}\sum_{\sigma\in\Pi(N)} m^\sigma(\underline{w})\right)\Box\left(\frac{1}{n!}\sum_{\sigma\in\Pi(N)} m^\sigma(\overline{w})\right) = \phi(\underline{w})\Box\phi(\overline{w}).$$

□

From Proposition 5.4 we obtain that $\Phi(w) = \Phi^\Box(w)$ for each $w \in CIG^N$. Now, we define $\mathcal{DC}^\Box(w) = DC(\underline{w})\Box DC(\overline{w})$ for each $w \in IH^N_{\mathcal{DC}}$ and notice that for convex interval games we have $\mathcal{DC}^\Box(w) = DC(\underline{w})\Box DC(\overline{w}) = C(\underline{w})\Box C(\overline{w}) = \mathcal{C}^\Box(w) = \mathcal{C}(w)$, where the second equality follows from the well known result in the theory of TU-games that for convex games the core and the dominance core coincide, and the last equality follows from Proposition 5.2.
Finally, we will show that the interval core is additive on the class of convex interval games, which is inspired by Dragan, Potters and Tijs (1989).

PROPOSITION 5.5. *The interval core $\mathcal{C} : CIG^N \to\to I(\mathbb{R})^N$ is an additive map.*

Proof. The interval core is a superadditive solution concept for all interval games (Alparslan, Branzei and Tijs (2008a)). We need to show the subadditivity of the interval core. We know by Theorem 5.2 that $\mathcal{C}(w) = \mathcal{W}^\Box(w)$ for $w \in CIG^N$. Take $w_1, w_2 \in CIG^N$. We have to prove that $\mathcal{C}(w_1 + w_2) \subset \mathcal{C}(w_1) + \mathcal{C}(w_2)$. Note that $m^\sigma(w_1 + w_2) = m^\sigma(w_1) + m^\sigma(w_2)$ for each $w_1, w_2 \in CIG^N$. By definition of the square interval Weber set we have $\mathcal{W}^\Box(w_1 + w_2) = W(\underline{w}_1 + \underline{w}_2)\Box W(\overline{w}_1 + \overline{w}_2)$. Hence, by Theorem 5.2, we obtain

$$\mathcal{C}(w_1 + w_2) = \mathcal{W}^\Box(w_1 + w_2) \subset \mathcal{W}^\Box(w_1) + \mathcal{W}^\Box(w_2) = \mathcal{C}(w_1) + \mathcal{C}(w_2).$$

□

6 Concluding remarks

In this paper we define and study convex interval games. We note that the combination of Theorems 3.1, 4.1 and 5.2 can be seen as an interval version of Theorem 1.1. In fact these theorems imply Theorem 1.1 for the embedded class of classical TU-games. There are still many interesting open questions. For further research it is interesting to solve the question whether one can extend to interval games the well known result in the traditional cooperative game theory that the core of a convex game is the unique stable set (Shapley (1971)). A thorough study of interval games arising from bankruptcy situations with interval data is work in progress by Alparslan Gök, Branzei and Tijs (2008b). It seems worthwhile to consider various Operations Research situations (Borm et al. (2001)) with interval data, where the corresponding interval games are convex or concave such as sequencing situations and airport situations. Also, we can try to extend to convex interval games the characterizations of classical convex games where exactness of subgames and superadditivity of marginal (or remainder) games play a role (Biswas et al. (1999), Branzei, Dimitrov and Tijs (2004) and Martinez-Legaz (2006)). It is also interesting to find an axiomatization of the interval Shapley value.

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